TWMS J. Pure Appl. Math., V.8, N.1, 2017, pp.83-96

STABILITY AND BIFURCATION ANALYSIS IN A DISCRETE-TIME PREDATOR-PREY DYNAMICS MODEL WITH FRACTIONAL ORDER

MOUSTAFA EL-SHAHED[1], A.M. AHMED[2], IBRAHIM M. E. ABDELSTAR[2,3]

Abstract. In this study, the dynamical behavior of a discrete predator–prey dynamics model of fractional-order is discussed. The existence conditions of the locally asymptotic stability and bifurcation theory of the equilibrium points are analyzed. The numerical simulations are carried out to illustrate the dynamical behaviors of the model such as flip bifurcation, Hopf bifurcation and chaos phenomenon. The results of numerical simulations verify our theoretical analysis.

Keywords: discrete time, predator–prey dynamics model, fractional order, bifurcation, chaos.

AMS Subject Classification: 39A30, 40A05, 92D25.

1. Introduction

A Predator is an animal that kills another to get food. The predator is usually larger and less number of prey and it must possess the ability to kill. Currently, there was a big interest in the dynamic characteristics of the predator-prey models. For example, Huang et al [20] has been investigating the issue of control of the bifurcation of the novel asymmetrical fractures predator-prey system with time delay, Xu [25] has studied the dynamics of a diffusive prey-predator model with general functional response and stage-structured for the prey. Ghaziani et al [16] have introduced the fractional order Leslie-Gower prey-predator model, which describes the interaction between two populations of prey and predator. In [20] Hu et al have investigated a discrete-time predator-prey system of Holling and Leslie type with a constant yield prey harvesting obtained from the forward Euler scheme.

The objectives of this paper are to study the dynamical behaviors of the discrete-time generalist predator–prey dynamics model with fractional order which was considered in [26]:

$$\begin{aligned} \frac{dx}{dt} &= x(1-x) - \frac{x^2 y}{1+ax^2}, \\ \frac{dy}{dt} &= \frac{bx^2 y}{1+ax^2} + \frac{cy}{1+dy} - ey. \end{aligned} \tag{1}$$

Where a, b, c, d and e are constant. Sufficient conditions for the existence of the solutions of the discrete-time model (1) with fractional-order is investigated. The equilibrium points and their asymptotic stability are discussed. Also, the conditions for the existence of flip bifurcation is derived. The necessary conditions for this system to exhibit chaotic dynamics are also derived.

[1]Department of Mathematics, Faculty of Arts and Sciences Qassim University, Qassim, Unizah, Saudi Arabia
[2]Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City, Cairo, Egypt
[3]Quantitative methods Unit, Faculty of Business & Economics Qassim University, Qassim, Buridah, Saudi Arabia
e-mail: elshahedm@yahoo.com, ahmedelkb@yahoo.com, essonny19@gmail.com

Manuscript received February 2016.

In recent decades, the fractional calculus and Fractional differential equations have attracted much attention and increasing interest due to their potential applications in science and engineering [24]. In this paper, we consider the fractional order model for generalist predator–prey dynamics model.

The purpose of this paper is to study the dynamical behaviors of the discrete-time generalist predator–prey dynamics model with fractional-order for the model (1). Sufficient conditions for the existence of the solutions of the discrete-time model (1) with fractional-order is investigated. The equilibrium points and their asymptotic stability are discussed. Also, the conditions for the existence of flip bifurcation is derived. The necessary conditions for this system to exhibit chaotic dynamics are also derived.

2. Fractional calculus

A great deal of research has been conducted on mathematical models based on fractional-order differential equations. A property of these fractional models is their nonlocal property which does not exist in integer order differential equations. Nonlocal property means that the next state of a model depends not only upon its current state, but also upon all of its historical status as the case in epidemics. Fractional-order differential equations can be used to model phenomena which cannot be adequately modeled by integer-order differential equations [17, 23, 24]. On the other hand, discrete-time models are more accurate to describe epidemics than the continuous-time models because statistical data on epidemics is collected in discrete time. Additional advantage includes simplicity of its simulation, the fact that is easier to adjust system's parameters from statistical data in discrete-time models than in continuous-time ones as well as the fact that discrete-time models may exhibit a richer dynamic behavior than its continuous-time counterparts. There are several definitions of fractional derivatives [2, 8]. One of the most common definition is the Caputo definition [9]. This definition is often used in real applications.

Definition 2.1. *The fractional integral of order $\beta \in \mathbb{R}^+$ of the function $f(t)$; $t > 0$; is defined by*

$$I^{\beta} f(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s) ds,$$

and the fractional derivative of order $\alpha \in (n-1, n)$ of $f(t)$; $t > 0$; is defined by

$$D_t^{\alpha} f(t) = I^{n-\alpha} D_t^n f^{(n)}(t), \ \alpha > 0,$$

where, $f^{(n)}$ represents the $n-$order derivative of $f(t)$; $n = [\alpha]$ is the value of α rounded up to the nearest integer, I^{β} is the $\beta-$order Riemann-Liouville integral operator and $\Gamma(.)$ is Euler's Gamma function. The operator D is called the " $\alpha-$order Caputo differential operator".

Although the Riemann–Liouville fractional derivative weakens the conditions on the function $f(t)$, Caputo's fractional derivatives are more widely used in initial values problems of differential equations and have stronger physical interpretations.

The fractional-order predator-prey dynamics model can be written as follows:[15]

$$\begin{aligned} D_t^{\alpha} x &= x(1-x) - \tfrac{x^2 y}{1+ax^2}, \\ D_t^{\alpha} y &= \tfrac{bx^2 y}{1+ax^2} + \tfrac{cy}{1+dy} - ey. \end{aligned} \tag{2}$$

where, D_t^{α} represents the Caputo fractional derivative, $t > 0$; and $\alpha \in (0, 1]$.

3. Discretization process

In [6, 7, 13, 14] a discretization method was introduced to discretize fractional order differential equations. We are interested in applying the discretizations method to the fractional predator–prey dynamics model (2). The discretization of system (2) with piecewise constant arguments is given as

$$\begin{aligned} D_t^\alpha x &= x\left(\left[\tfrac{t}{h}\right]h\right)\left[1 - x\left(\left[\tfrac{t}{h}\right]h\right)\right] - \tfrac{x^2\left(\left[\frac{t}{h}\right]h\right)y\left(\left[\frac{t}{h}\right]h\right)}{1+ax^2\left(\left[\frac{t}{h}\right]h\right)}, \\ D_t^\alpha y &= \tfrac{bx^2\left(\left[\frac{t}{h}\right]h\right)y\left(\left[\frac{t}{h}\right]h\right)}{1+ax^2\left(\left[\frac{t}{h}\right]h\right)} + \tfrac{cy\left(\left[\frac{t}{h}\right]h\right)}{1+dy\left(\left[\frac{t}{h}\right]h\right)} - ey\left(\left[\tfrac{t}{h}\right]h\right). \end{aligned} \tag{3}$$

First, let t satisfies the inequality $0 \le t < h$; which implies that $0 \le \frac{t}{h} < 1$. Then, we get

$$\begin{aligned} D_t^\alpha x_1 &= x_0\left[1 - x_0\right] - \tfrac{x_0^2 y_0}{1+ax_0^2}, \\ D_t^\alpha y_1 &= \tfrac{bx_0^2 y_0}{1+ax_0^2} + \tfrac{cy_0}{1+dy_0} - ey_0. \end{aligned} \tag{4}$$

The solution of (4) reduces to

$$\begin{aligned} x_1(t) &= x_0 + J^\alpha\left[x_0\left(1 - x_0\right) - \tfrac{x_0^2 y_0}{1+ax_0^2}\right], \\ y_1(t) &= y_0 + J^\alpha\left[\tfrac{bx_0^2 y_0}{1+ax_0^2} + \tfrac{cy_0}{1+dy_0} - ey_0\right]. \end{aligned} \tag{5}$$

where $J^\alpha = \frac{h^\alpha}{\Gamma(1+\alpha)}$. Second, let $t \in [h, 2h)$, which makes $1 \le \frac{t}{h} < 2$. Thus, we obtain

$$\begin{aligned} D_t^\alpha x_2 &= x_1\left(1 - x_1\right) - \tfrac{x_1^2 y_1}{1+ax_1^2}, \\ D_t^\alpha y_2 &= \tfrac{bx_1^2 y_1}{1+ax_1^2} + \tfrac{cy_1}{1+dy_1} - ey_1, \end{aligned} \tag{6}$$

which has the following solution:

$$\begin{aligned} x_2(t) &= x_1 + \tfrac{(t-h)^\alpha}{\Gamma(1+\alpha)}\left[x_1\left(1 - x_1\right) - \tfrac{x_1^2 y_1}{1+ax_1^2}\right], \\ y_2(t) &= y_1 + \tfrac{(t-h)^\alpha}{\Gamma(1+\alpha)}\left[\tfrac{bx_1^2 y_1}{1+ax_1^2} + \tfrac{cy_1}{1+dy_1} - ey_1\right]. \end{aligned} \tag{7}$$

Repeating the discretization process n times yields

$$\begin{aligned} x_{n+1}(t) &= x_n(nh) + \tfrac{(t-nh)^\alpha}{\Gamma(1+\alpha)}\left[x(nh)\left[1 - x(nh)\right] - \tfrac{x^2(nh)y(nh)}{1+ax^2(ns)}\right], \\ y_{n+1}(t) &= y_n(nh) + \tfrac{(t-nh)^\alpha}{\Gamma(1+\alpha)}\left[\tfrac{bx^2(ns)y(ns)}{1+ax^2(ns)} + \tfrac{cy(ns)}{1+dy(ns)} - ey(ns)\right], \end{aligned} \tag{8}$$

where, $t \in [nh, (n+1)h)$: For $t \to (n+1)h$, system (8) is reduced to

$$\begin{aligned} x_{n+1} &= x_n + \tfrac{h^\alpha}{\Gamma(1+\alpha)}\left[x_n\left(1 - x_n\right) - \tfrac{x_n^2 y_n}{1+ax_n^2}\right], \\ y_{n+1} &= y_n + \tfrac{h^\alpha}{\Gamma(1+\alpha)}\left[\tfrac{bx_n^2 y_n}{1+ax_n^2} + \tfrac{cy_n}{1+dy_n} - ey_n\right]. \end{aligned} \tag{9}$$

Remark 1. If the fractional parameter α tends to one in Eq. (9), then we have the forward Euler discretization of system (2).

In the following, we will study the dynamics of the system (9).

4. Stability of equilibria

In this section, we discuss the stability [3-5] of the discrete generalist predator–prey dynamics model with fractional-order (9). First, we need the following lemma.

Lemma 4.1. [10, 11, 12]*: Let λ_1 and λ_2 be the two roots of the characteristic equation of Jacobian matrix $J\left[E^*(x^*, y^*)\right]$, which are called eigenvalues of the fixed point (x^*, y^*), then we have the following definitions:*

(i) A fixed point E^* is called a sink if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, so the sink is locally asymptotically stable.

(ii) A fixed point E^* is called a source if $|\lambda_1| > 1$ and $|\lambda_2| > 1$, so the source is locally unstable.

(iii) A fixed point E^* is called a saddle if $|\lambda_1| > 1$ and $|\lambda_2| < 1$ (or $|\lambda_1| < 1$ and $|\lambda_2| > 1$).

(iv) A fixed point E^* is called non-hyperbolic if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

To evaluate the equilibrium points we solve (9), then

1. The trivial state $E_0 = (0, 0)$. The point E_0 always exists.

2. The second semi-trivial equilibrium point is $E_1 = (x_1, y_1) = (1, 0)$ when the predator is absent in the prey, in this case $(y = 0)$, therefore the prey is not an exhibition of predation. The point E_1 always exists.

3. The third semi-trivial equilibrium point is $E_2 = \left(0, \frac{b}{ed(1+a)R_0}\right)$ which is exist if $R_0 > 0$.

4. The last one is the interior equilibrium point $E_3 = (x_*, y_*)$, where x_* is the real root of the equation

$$f(Z) = ad(ae-b)\left(Z^5 - Z^4\right) - \left[(ae-b)(d-1)+\right.$$

$$\left.a(de+c)\right]Z^3 - d(2ae-b)Z^2 + \left[e\,(d-1)+c\right]Z - de = 0,$$

and $y_3 = \frac{(1-x_3)\left(1+ax_3^2\right)}{x_3}$, E_3 must be having non negative component, then we have the condition $0 < x_3 < 1$. By the simple computation, we can see that the basic reproductive number is $R_0 = \frac{b}{(1+a)(c-e)}$.

Now, the Jacobian matrix $J\left(E_0\right)$ for a system given in (2.2) evaluated at E_0 is as follows:

$$J\left(E_0\right) = \begin{pmatrix} 1 + \frac{h^\alpha}{\Gamma(1+\alpha)} & 0 \\ 0 & 1 + \frac{h^\alpha b}{\Gamma(1+\alpha)(1+a)R_0} \end{pmatrix}. \tag{10}$$

Theorem 4.1. *The trivial-equilibrium point E_0 has at least two different topological types of its all values of parameters*

(i) E_0 is a source if $R_0 < 0$ or $R_0 > \frac{2bh^\alpha}{(1+a)\Gamma(1+\alpha)}$,

(ii) E_0 is a saddle if $0 < h < \sqrt[\alpha]{\frac{2(1+a)\Gamma(1+\alpha)R_0}{b}}$, $R_0 > 0$.

Proof. The eigenvalues corresponding to the equilibrium point E_0 are $\lambda_{01} = 1 + \frac{h^\alpha}{\Gamma(1+\alpha)} > 0$ and $\lambda_{02} = 1 + \frac{h^\alpha b}{\Gamma(1+\alpha)(1+a)R_0}$, where $\alpha \in (0, 1]$ and $h, \frac{h^\alpha}{\Gamma(1+\alpha)} > 0$. Hence, applying the stability conditions using lemma 4.1, one can obtain the results *(i), (ii)*.

To investigate the stability of E_1, let $J\left(E_1\right)$ is the Jacobian matrix for a system given in (9) evaluated at E_1 then:

$$J\left(E_1\right) = \begin{pmatrix} 1 - \frac{h^\alpha}{\Gamma(1+\alpha)} & -\frac{h^\alpha}{\Gamma(1+\alpha)(1+a)} \\ 0 & 1 + \frac{h^\alpha b}{\Gamma(1+\alpha)(1+a)}\left(\frac{1}{R_0} + 1\right) \end{pmatrix}. \tag{11}$$

□

Theorem 4.2. *The semi-trivial equilibrium point E_1 has at least four different topological types of its all values of parameters (i) E_1 is a sink if* $0 < h < \min\left\{ \sqrt[\alpha]{2\Gamma(1+\alpha)}, \sqrt[\alpha]{\frac{-2(1+a)\Gamma(1+\alpha)R_0}{b(R_0+1)}} \right\}$, $-1 < R_0 < 0$,

(ii) E_1 is a source if $\sqrt[\alpha]{2\Gamma(1+\alpha)} < h < \sqrt[\alpha]{\frac{-2(1+a)\Gamma(1+\alpha)R_0}{b(R_0+1)}}$, $R_0 < -1$,

(iii) E_1 is a saddle if $0 < h < \min\left\{ \sqrt[\alpha]{2\Gamma(1+\alpha)}, \sqrt[\alpha]{\frac{-2(1+a)\Gamma(1+\alpha)R_0}{b(R_0+1)}} \right\}$, $R_0 < -1$ *or* $\sqrt[\alpha]{2\Gamma(1+\alpha)} < h < \sqrt[\alpha]{\frac{-2(1+a)\Gamma(1+\alpha)R_0}{b(R_0+1)}}$, $-1 < R_0 < 0$,

(iv) E_1 is non-hyperbolic if $h = \sqrt[\alpha]{2\Gamma(1+\alpha)}$ *or* $h = \sqrt[\alpha]{\frac{-2(1+a)\Gamma(1+\alpha)R_0}{b(R_0+1)}}$, $R_0 \neq -1$.

Proof. The eigenvalues corresponding to the equilibrium point E_1 are $\lambda_{11} = 1 - \frac{h^\alpha}{\Gamma(1+\alpha)}$ and $\lambda_{12} = 1 + \frac{h^\alpha b}{\Gamma(1+\alpha)(1+a)}\left(\frac{1}{R_0}+1\right)$, where $\alpha \in (0,1]$. Hence, applying the stability conditions [3, 4] using lemma 4.1, one can obtain the results *(i)-(iv)*. □

Theorem 4.3. *The semi-trivial equilibrium point E_1 loses its stability via a flip point where*

$$h = \sqrt[\alpha]{2\Gamma(1+\alpha)} \ \textit{or}\ h = \sqrt[\alpha]{\frac{-2(1+a)\Gamma(1+\alpha)R_0}{b(R_0+1)}}, R_0 \neq -1.$$

Proof. The flip bifurcation occurs when one of the eigenvalues of $J(E_1)$ equal to -1. This local bifurcation entails the birth of a period 2-cycle. Hence, the system (9) may undergo a flip bifurcation at E_1 where

$$h = \sqrt[\alpha]{2\Gamma(1+\alpha)} \text{ or } h = \sqrt[\alpha]{\frac{-2(1+a)\Gamma(1+\alpha)R_0}{b(R_0+1)}}, R_0 \neq -1.$$

□

To investigate the stability of E_2, let $J(E_2)$ is the Jacobian matrix for a system given in (9) evaluated at E_2, then:

$$J(E_2) = \begin{pmatrix} 1 + \frac{h^\alpha}{\Gamma(1+\alpha)} & 0 \\ 0 & 1 - \frac{bceR_0(1+a)h^\alpha}{\Gamma(1+\alpha)[eR_0(1+a)+b]^2} \end{pmatrix}. \tag{12}$$

Theorem 4.4. *The semi-trivial equilibrium point E_2 has at least four different topological types of its all values of parameters*

(i) E_2 is a source if $h > \sqrt[\alpha]{\frac{2\Gamma(1+\alpha)[eR_0(1+a)+b]^2}{bceR_0(1+a)}}$, $R_0 > 0$ *or* $R_0 < 0$,

(ii) E_2 is a saddle if $0 < h < \sqrt[\alpha]{\frac{\Gamma(1+\alpha)[eR_0(1+a)+b]^2}{bceR_0(1+a)}}$, $R_0 > 0$.

Proof. The eigenvalues corresponding to the equilibrium point E_2 are $\lambda_{21} = 1 + \frac{h^\alpha}{\Gamma(1+\alpha)} > 0$ and $\lambda_{22} = 1 - \frac{bceR_0(1+a)h^\alpha}{\Gamma(1+\alpha)[eR_0(1+a)+b]^2}$, where $\alpha \in (0,1]$. Hence, applying the stability conditions using lemma 4.1, one can obtain the results *(i), (ii)*. □

For the dynamical properties of the positive equilibrium point E_3 we need to state this lemma.

Lemma 4.2. [1, 18, 19, 21, 22, 26]*: Let $F(\lambda) = \lambda^2 - Tr\lambda + Det$. Suppose that $F(1) > 0$, λ_1 and λ_2 are the two roots of $F(\lambda) = 0$. Then,*

(i) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $Det < 1$,

(ii) $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$) if and only if $F(-1) < 0$,

(iii) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$ and $Det > 1$,

(iv) $\lambda_1 = -1$ and $\lambda_2 \neq 1$ if and only if $F(-1) = 0$ and $Tr \neq 0, 2$,

(v) λ_1 and λ_2 are complex and $|\lambda_1| = |\lambda_2|$ if and only if $Tr^2 - 4Det < 0$ and $Det = 1$.

The necessary and sufficient conditions ensuring that $|\lambda_1| < 1$ and $|\lambda_2| < 1$ are:[1]

$$\begin{array}{l}(i)\, 1 - TrJ + \det J > 0,\\ (ii)\, 1 + TrJ + \det J > 0,\\ (iii)\, \det J < 1.\end{array} \tag{13}$$

If one of the conditions (13) is not satisfied, then we have one of the following cases:[12]

1. A saddle-node (often called fold bifurcation in maps), transcritical or pitchfork bifurcation if one of the eigenvalues $= 1$ and other eigenvalues (real) $\neq 1$. This local bifurcation leads to the stability switching between two different steady states;

2. a flip bifurcation if one of the eigenvalues $= -1$, other eigenvalues (real) $\neq -1$. This local bifurcation entails the birth of a period 2-cycle;

3. a Neimark–Sacker (secondary Hopf) bifurcation in this case we have two conjugate eigenvalue and the modulus of each of them $= 1$.

This local bifurcation implies the birth of an invariant curve in the phase plane. Neimark–Sacker bifurcation is considered an equivalent to the Hopf bifurcation in continuous time and in fact the major instrument to prove the existence of quasi-periodic orbits for the map.

Note. one can get any local bifurcation (fold, flip and Neimark–Sacker) by taking specific parameter value such that one of the condition of each bifurcation satisfied.

The Jacobian matrix $J(E_3)$ for a system given in (9) evaluated at E_3 is as follows:

$$J(E_3) = \begin{pmatrix} 1 + \frac{h^\alpha}{\Gamma(1+\alpha)}\left[\frac{2ax_3^3y_3}{\left(ax_3^2+1\right)^2} - 1\right] & -\frac{h^\alpha x_3^2}{\Gamma(1+\alpha)\left(ax_3^2+1\right)} \\ \frac{2bh^\alpha x_3y_3}{\Gamma(1+\alpha)\left(ax_3^2+1\right)}\left(1 - \frac{ax_3^2}{ax_3^2+1}\right) & 1 - \frac{cdh^\alpha y_3}{\Gamma(1+\alpha)(dy_3+1)^2} \end{pmatrix}, \tag{14}$$

then we have

$$Tr = trace\left[J(E_3)\right] = \frac{Ah^\alpha}{\Gamma(1+\alpha)} + 2,$$

and

$$Det = \det\left[J(E_3)\right] = \frac{h^\alpha}{\Gamma(1+\alpha)}\left[B\left(\frac{h^\alpha}{\Gamma(1+\alpha)}\right) - C\right] + 1,$$

where

$$A = \frac{2ax_3^3y_3}{\left(ax_3^2+1\right)^2} - \frac{cdy_3}{(dy_3+1)^2} - 1,$$

$$\begin{aligned} B &= \frac{1}{\left(ax_3^2+1\right)^3(dy_3+1)^2}\left[y_3(a^3x_3^6cd + (2by_3^2(a-1)d^2 - \right. \\ &\quad 2y_3(2b - 2ba + a^2c)d + 2b(a-1))x_3^5 + 3a^2x_3^4cd + \\ &\quad \left.(2by_3^2d^2 - 2y_3(ac-2b)d + 2b)x_3^3 + 3ax_3^2cd + cd)\right], \end{aligned}$$

$$\begin{aligned} C &= \frac{1}{\left(ax_3^2+1\right)^2(dy_3+1)^2}(x_3^4(1 + d^2y_3^2 + d(c+2)y_3)a^2 + \\ &\quad 2x_3^2(1 - y_3(dy_3+1)^2x_3 + d^2y_3^2 + d(c+2)y_3)a + 1 + \\ &\quad d^2y_3^2 + d(c+2)y_3), \end{aligned}$$

Then, we get the following result.

Theorem 4.5. *If* $B > 0$ *we have*

(i) E_3 *is asymptotically stable (sink) if one of the following conditions holds:*

(i.1) $C \geq A + 4\sqrt{B}$ *and* $0 < h < h_1$.

(i.2) $C < A + 4\sqrt{B}$ *and* $0 < h < h_2$.
(ii) E_3 *is unstable (source) if one of the following conditions holds:*
(ii.1) $C \geq A + 4\sqrt{B}$ *and* $h > h_3$.
(ii.2) $C < A + 4\sqrt{B}$ *and* $h > h_2$..
(iii) E_3 *is unstable (saddle) if* $C \geq A + 4\sqrt{B}$ *and* $h_1 < h < h_3$.
(iv) E_3 *is non-hyperbolic if one of the following conditions holds:*
(iv.1) $C \geq A + 4\sqrt{B}$ *and* $h = h_1$ *or* h_3.
(iv.2) $C < A + 4\sqrt{B}$ *and* $h = h_2$. *where*

$$h_1 = \sqrt[\alpha]{\frac{\Gamma(1+\alpha)\left[C - A - \sqrt{\Delta}\right]}{2B}}, \; h_3 = \sqrt[\alpha]{\frac{\Gamma(1+\alpha)\left(C - A + \sqrt{\Delta}\right)}{2B}},$$

$$h_2 = \sqrt[\alpha]{\frac{C\Gamma(1+\alpha)}{B}} \text{ and } \Delta = (C-A)^2 - 16B.$$

Proof. By applying lemma 4.6 and the Jury's conditions [22], we can easily get the stability conditions *(i)-(iv).* □

Theorem 4.6. *If* $B > 0$ *then* E_3 *loses its stability via*
(i) flip bifurcation when $C \geq A + 4\sqrt{B}$ *and* $h = h_1$ *or* $h = h_3$.
(ii) Hopf bifurcation when $C < A + 4\sqrt{B}$ *and* $h = h_2$.

Proof. [14] introduced a Thorough study of the main types of bifurcations for 2-D maps. In line with this study, we can see that E_i undergoes flip bifurcation when a single eigenvalue becomes equal to -1 (i.e. $1 - Tr + Det = 0$). Therefore E_3 can lose its stability through flip bifurcation when $h = h_1$ or $h = h_3$. A flip bifurcation of E_3 may occur if the parameters vary in the small neighborhood of the following sets

$$S_1 = \left\{(h, \alpha, a, b, c, d, e) : C \geq A + 4\sqrt{B} \text{ and } h = h_1\right\},$$

or

$$S_2 = \left\{(h, \alpha, a, b, c, d, e) : C \geq A + 4\sqrt{B} \text{ and } h = h_3\right\}.$$

When the Jacobian has pair of complex conjugate eigenvalues of modulus 1, we get the Hopf bifurcation (i.e. $Det = 1$), then E_3 can lose its stability through Hopf bifurcation when $h = h_2$ and then it also implies that all the parameters locate and vary in the small neighborhood of the following set:

$$S_3 = \left\{(h, \alpha, a, b, c, d, e) : C < A + 4\sqrt{B} \text{ and } h = h_2\right\}.$$

□

It can be concluded from the statements of Theorem 4.7 that the positive equilibrium is locally asymptotically stable if and only if the fractional parameter $h \in (0, h^*)$ where $h^* = h_1$ or h_2 or h_3 (according $C \geq A + 4\sqrt{B}$ or $C < A + 4\sqrt{B}$). As increases above the critical value h^*, the positive equilibrium is unstable, and a limit cycle is expected to appear in the proximity of E_3, due to the Flip bifurcation phenomenon.

5. Numerical simulations

In this section, we give the phase portraits, the attractor of parameter h and bifurcation diagrams to confirm the above theoretical analysis and to obtain more dynamical behaviors of generalist predator–prey dynamics model (9). Since most of the fractional-order differential equations do not have exact analytic solutions, approximation and numerical techniques must be used.

From the numerical results Figures follows, it is clear that the approximate solutions depend on the fractional parameters h, α see fig. (1). The approximate solutions x_n and y_n are displayed in all figures below.

First, we will study the phase portrait of the model (9) in figure (1), we use some documented data for some parameters like $a = 100$, $b = 80$, $c = 1$, $d = 0.9$ and $e = 0.5$, other parameter will be (a) $h = 0.12$, $\alpha = 0.9$ and $(x_1, y_1) = (0.1, 35)$ and $(0.2, 35)$, (b) $h = 0.05$, $\alpha = 0.9$ and $(x_1, y_1) = (0.1, 35)$ and $(0.2, 35)$, (c) $h = 0.17$, $\alpha = 0.9$ and $(x_1, y_1) = (0.1, 35)$ and $(0.2, 35)$, (d) $h = 0.05$, $\alpha = 0.7$ and $(x_1, y_1) = (0.11, 17.9)$, (e) $h = 0.05$, $\alpha = 1.9$ and $(x_1, y_1) = (0.11, 17.9)$, (f) $h = 0.05$, $\alpha = 2.6$ and $(x_1, y_1) = (0.11, 17.9)$, .

Fig. 1 depicts the phase portraits of model (9) according to the chosen parameter values and for various values of the fractional-order parameters h and α. We can see that, whenever fixed the value of α and increased the value of h then E_3 moves from the stabilized to the chaotic band. Fig. (1c) depict the phase portrait for model (9) which exist before a flip bifurcation for $h < h^*$ and the phase portrait which exist after a flip for $h > h^*$.

By computing, we have $E_2 = (0.11097, 17.877)$, $C \approx 0.074$, $A + 4\sqrt{B} \approx 2.3 \Longrightarrow C < A + 4\sqrt{B}$and we can get the critical value of flip bifurcation for model (9) s. t.

In figs. 1(a)-1(c) we have $h^* = h_2 = 0.1684935728$.

In figs. 1(d)-1(f) we have $h^* = h_2 =$ (d) 0.0934, (e) 0.603 and (f) 0.908 according to α. Thus, according to the theorem (4.7), the conditions of flip bifurcation are achieved near the positive equilibrium E_3, flip bifurcation diagrams for the parameter h is plotted in Fig. (2).

In figure (2) we use some documented data for some parameters like $\alpha = 0.95$, $a = 100$, $b = 40$, $c = 1$, $d = 0.9$, $e = 0.5$ and $(x_1, y_1) = (0.9, 9.5)$, other parameter will be (a) $h = 2.5$, (b) $h = 2.8$, (c) $h = 3.25$, (d) $h = 3.32$, (e) $h = 3.34$, (f) $h = 3.35$ and (g) $h = 2.5 : 3.6$.

Fig. 2(a) describes the stable equilibrium of the model (9) according to the values of the parameters set out above, Period-2,4,8,16 orbits appear in Fig. (2b-2e). From Fig. (2f,2g) we can see that reducing and increasing h cause the disappearance of 2n-periodic orbits and increasing the chaotic attractors and the results of numerical simulation verify our theoretical analysis.

In figure (3) we use some documented data for some parameters like $h = 2.6$, $\alpha = 0.75 : 2.1$, $a = 100$, $b = 40$, $c = 1$, $d = 0.9$, $e = 0.5$ and $(x_1, y_1) = (0.9, 9.5)$.

In figure (4) we use some documented data for some parameters like $h = 0.5$, $\alpha = 0.95$, $a = 42 : 65$, $b = 40$, $c = 1$, $d = 0.9$, $e = 0.5$ and $(x_1, y_1) = (0.9, 9.5)$.

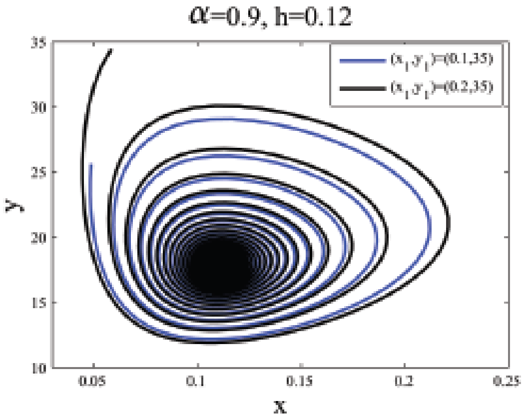
In figure (5) we use some documented data for some parameters like $h = 0.5$, $\alpha = 0.95$, $a = 100$, $b = 47.6 : 84.5$, $c = 1$, $d = 0.9$, $e = 0.5$ and $(x_1, y_1) = (0.9, 9.5)$.

In figure (6) we use some documented data for some parameters like $h = 0.5$, $\alpha = 0.95$, $a = 100$, $b = 40$, $c = 2.87 : 2.97$, $d = 0.9$, $e = 0.5$ and $(x_1, y_1) = (0.9, 9.5)$.

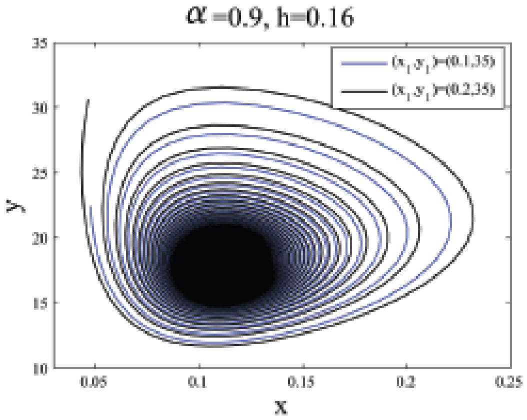
In figure (7) we use some documented data for some parameters like $h = 0.5$, $\alpha = 0.95$, $a = 100$, $b = 40$, $c = 1$, $d = 0.27 : 0.29$, $e = 0.5$ and $(x_1, y_1) = (0.9, 9.5)$.

In figure (8) we use some documented data for some parameters like $h = 0.5$, $\alpha = 0.95$, $a = 100$, $b = 40$, $c = 1$, $d = 0.9$, $e = 0.416 : 428$ and $(x_1, y_1) = (0.9, 9.5)$.

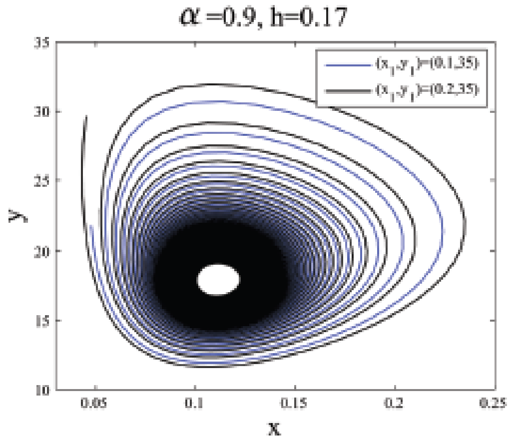
Figures (1) - (8) prove that each parameter of (9) is the bifurcation parameter model, which means that it may be any slight parameter change can effect in the stability of an equilibrium point.



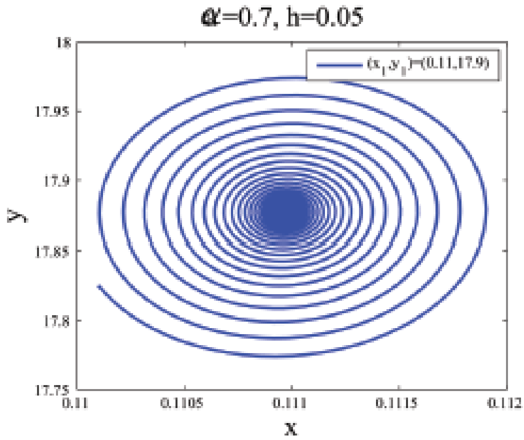
(a) When α = 0.9, h = 0.12 the trajectory of system (3.7) converges to E3.



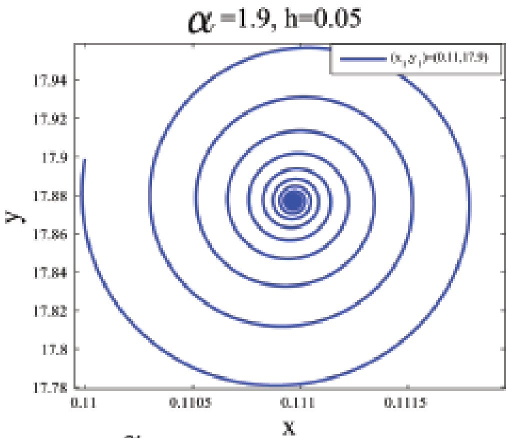
(b) When α = 0.9, h = 0.16 the trajectory of system (3.7) converges to E3.



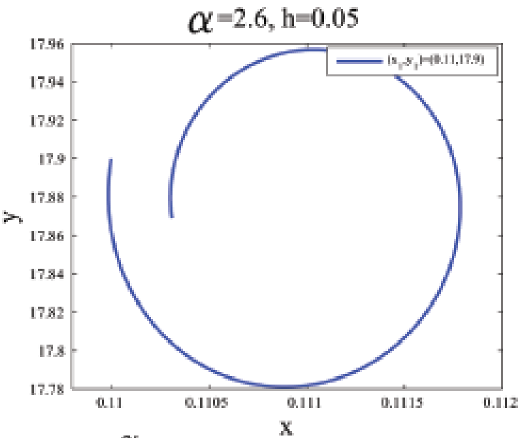
(d) When α = 0.9, h = 0.17 the trajectory of system (3.7) converges to an asymptotically stable limit cycle around E3.



(d) When α = 0.7, h = 0.05 the trajectory of system (3.7) converges to E3.



(e) When α = 1.9, h = 0.05 the trajectory of system (3.7) converges to E3.



(f) When α= 2:6; h = 0:05 the trajectory of system (3.7) converges slowly to E3.

Figure 1. Phase portraits of model (3.7).

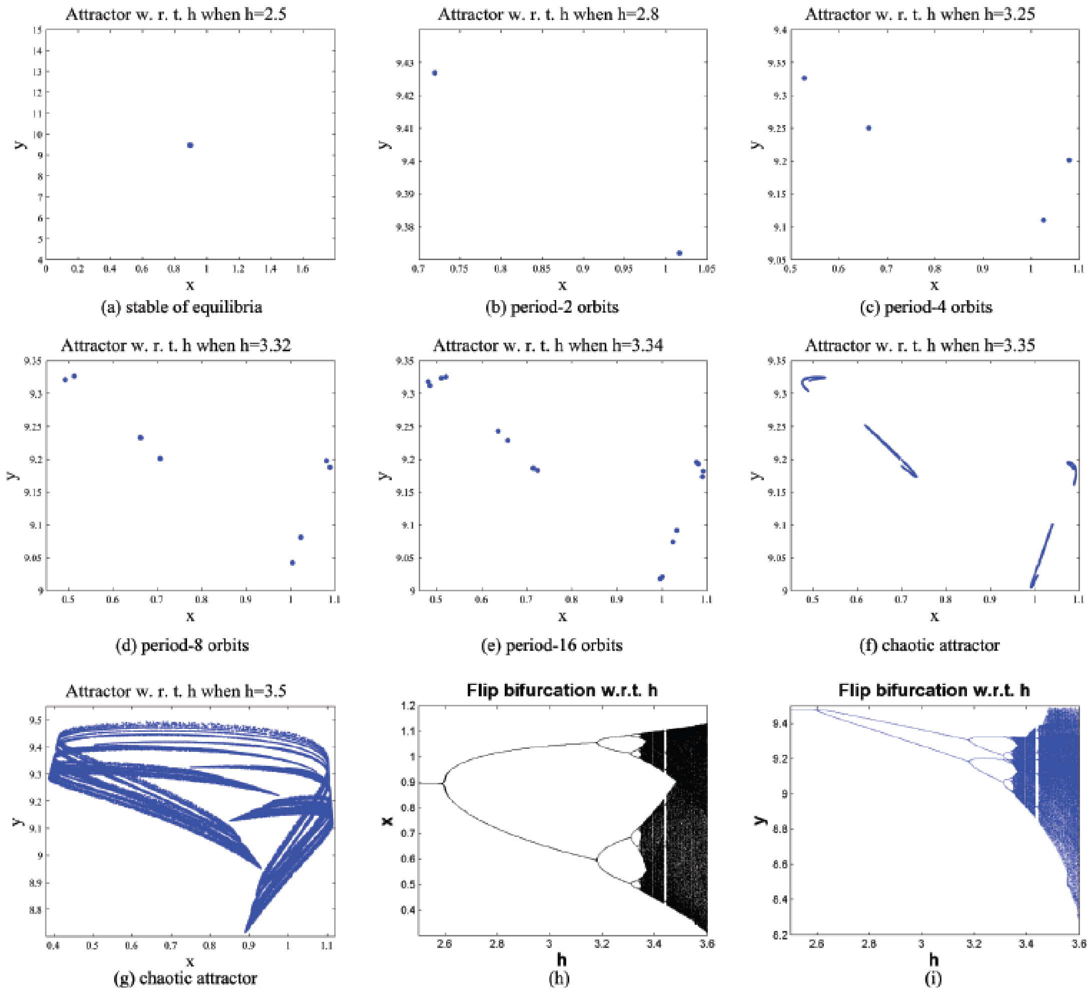


Figure 2.

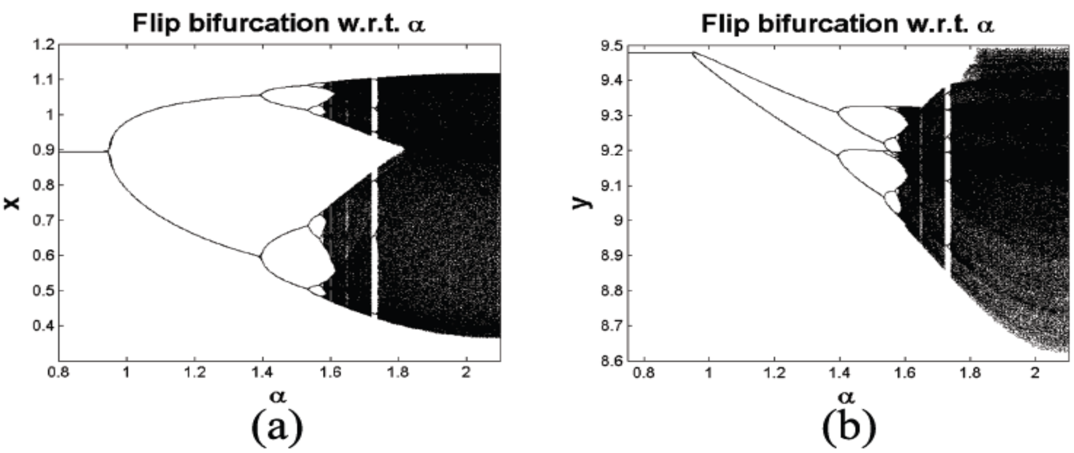


Figure 3.

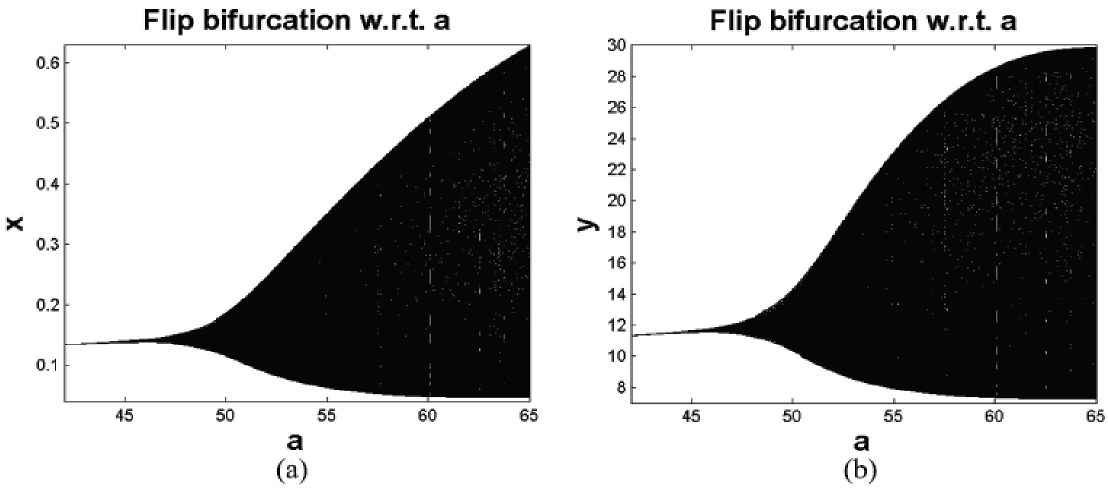


Figure 4.

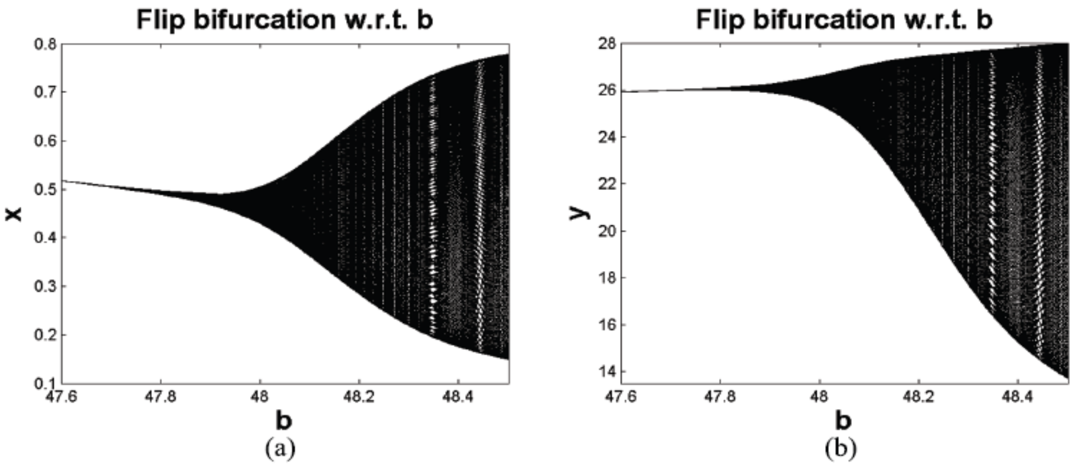


Figure 5.

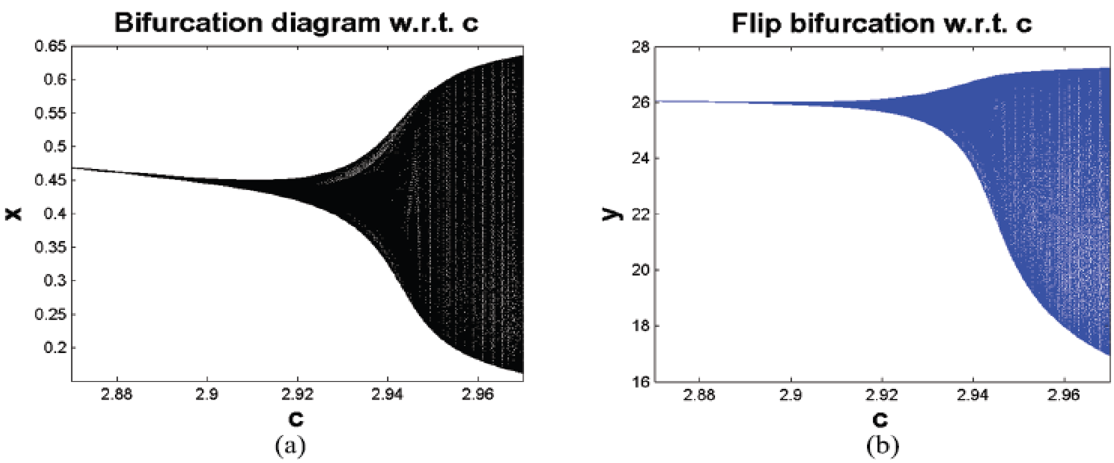


Figure 6.

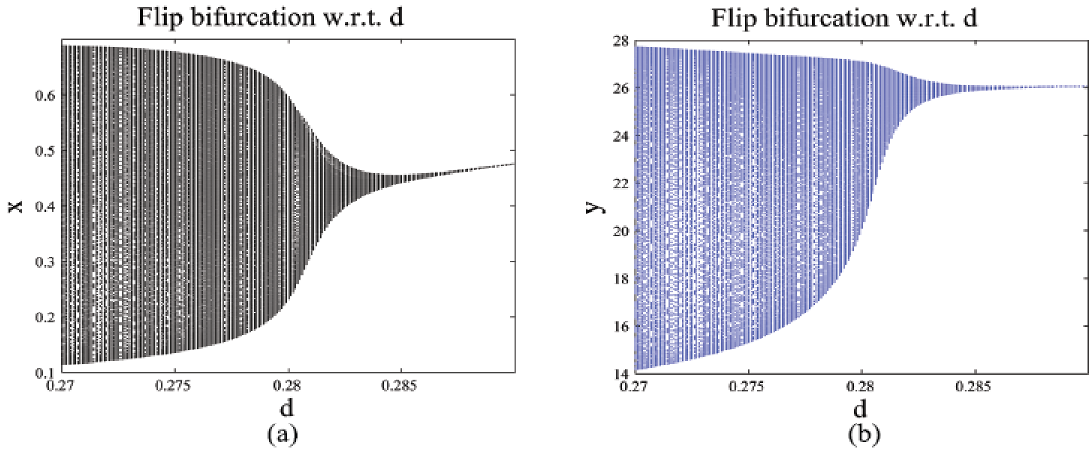


Figure 7.

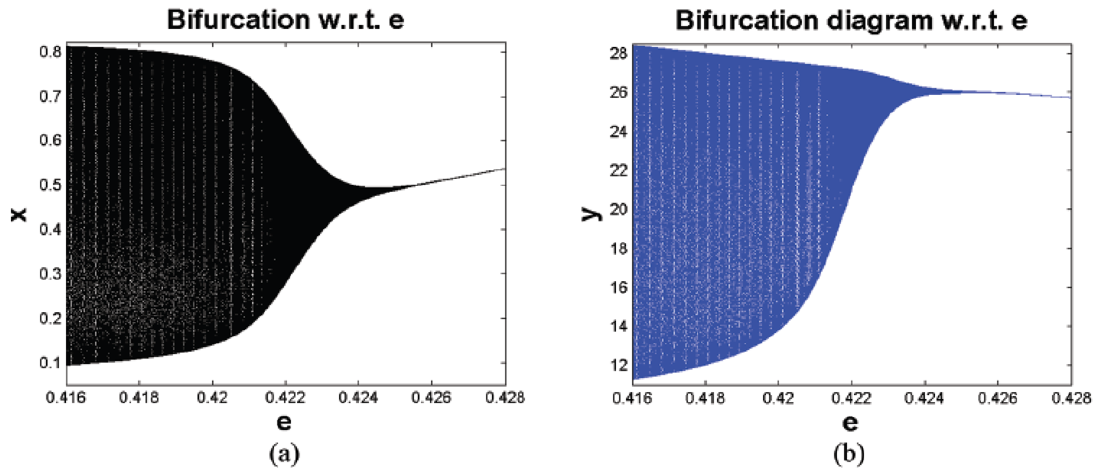


Figure 8.

6. Conclusions

In this paper we have introduced discrete generalist predator–prey dynamics model (9) with fractional-order. We have investigated the existence of the free and positive equilibriums. We have also studied the stability of the four equilibrium states of model (9). The flip and Hopf bifurcations are also investigated for the model (9) for some parameters like a, b, c, d and e the fractional-order parameters h and α. Also, a flip bifurcation process with respect to the fractional parameter h has been done to show the transformation of a classical model into a fractional one makes it very sensitive to the order of differentiation α and the fractional parameter h (see fig. (1)). It has been found that fractional-order parameters h and have obvious effect on the stability of the model (9). It has also been shown that the combining between the fractional and discrete dynamical models exhibit these models much richer dynamics and give more control on the model (9). Numerical simulations have been used to show the chaotic attractors of model (9).

Moreover, we noticed all the parameters affect the occurrence of flip bifurcation and chaos (see figs. (2), (3)), Hopf bifurcation in figs. (4)-(8).

References

[1] Agiza, H.N., Elabbasy, E.M., El-Metwally, H., Elsadany, A.A., (2009), Chaotic dynamics of a discrete prey-predator model with Holling type II. Nonlinear Analysis: Real World Applications, 10 pp.116-129.

[2] Ahmad, W.M., Sprott, J.C., (2003), Chaos in fractional-order autonomous nonlinear systems, Chaos, Solitons and Fractals, 16, pp.339–351.

[3] Aliev, F.A., Larin, V.B., (1980), Numerical-solution of discrete algebraic Riccati equation, Izvestiya Akademii Nauk Azerbaidzhanskoi SSR Seriya Fiziko-Tekhnicheskikh I Matematicheskikh Nauk, 5, pp.94-104.

[4] Aliev, F.A., Bordiug, B.A., Larin, V.B., (1987), A spectral method for solving Riccati matrix algebraic equations, Doklady Akademii Nauk SSSR, 292(4), pp.783-788.

[5] Aliev, F.A., Larin, V.B., (2014), On the algorithms for solving discrete periodic Riccati equation, Applied and Computational Mathematics, 13 (1), pp.46-54.

[6] Babakhani A., (2015), On the existence of solutions in coupled system of non-linear fractional integro-differential equations on the half line, TWMS J. Pure Appl. Math., 6(1), pp.48-58.

[7] Barbosa, R.S., Machado, J., Vinagre, B.M., Calderon, A.J., (2007), Analysis of the Van der Pol oscillator containing derivatives of fractional order, J. Vib. Control, 13 , pp.1291–1301.

[8] Caputo, M., (1967), Linear models of dissipation whose Q is almost frequency independent: II Geophys. J. R. Astron. Soc., 13, pp.529–539.

[9] Duan,J.S., (2013), The periodic solution of fractional oscillation equation with periodic input, Advances in Mathematical Physics, Article ID 869484, pp.1-6.

[10] El-Sayed, A.M.A., Salman, S.M., (2013), On a discretization process of fractional order Riccati differential equation, J. Fract. Calc. Appl., 4 , pp.251-259.

[11] El-Sayed, A.M.A., Salman, S.M., (2013), Fractional-order Chua's system: discretization, bifurcation and chaos, Adv. in Diff. Eqs. Springer, pp.1-13.

[12] Elsadany, A.A., Matouk, A.E., (2015), Dynamical behaviors of fractional-order Lotka-Voltera predator-prey model and its discretization, J. Appl. Math. Comput., 49, pp.269-283.

[13] El-Shahed, M., Ahmed, A.M., Abdelstar, I.M.E., (2017), Dynamics of a Plant-Herbivore Model with Fractional Order, Progr. Fract. Differ. Appl., 3(1), pp.59-67.

[14] Elaydi, S., (2008), Discrete Chaos with Applications in Science and Engineering. 2nd edn. Chapman and Hall/CRC, Boca Raton.

[15] Erbach, A., Lutscher, F., Seo, G., (2013), Bistability and limit cycles in generalist predator-prey dynamics, Ecological Complexity, 14, pp.48–55.

[16] Ghaziani, R.K., Alidoustia, J., Bayati Eshkaftakib, A., (2016), Stability and dynamics of a fractional order Leslie–Gower prey–predator model, Applied Mathematical Modelling, 40, pp.2075–2086.

[17] Hegazi, A.S., Ahmed, E., Matouk, A.E., (2011), The effect of fractional order on synchronization of two fractional order chaotic and hyperchaotic systems, Journal of Fractional Calculus and Applications, 1, pp.1-15.

[18] Z. Hu, Z. Teng, L. Zhang, (2014), Stability and bifurcation in a discrete SIR epidemic model, Math. Comput. in Simul, 97, pp.80-93.

[19] Hu, Z., Teng, Z., Zhang, L., (2011), Stability and bifurcation analysis of a discrete predator–prey model with non-monotonic functional response, Nonlinear Analysis: Real World Applications, 12, pp.2356–2377.

[20] D. Hu, H. Cao, (2015), Bifurcation and chaos in a discrete-time predator–prey system of Holling and Leslie type, Commun Nonlinear Sci Numer Simulat, 22, pp.702–715.

[21] Huang, J., Ruan, S., Song, J., (2014), Bifurcations in a predator–prey system of Leslie type with generalized Holling type III functional response, Journal of Differential Equations, 257, pp.1721–1752.

[22] Jury, E.I., (1974), Inner and stability of dynamic systems, New York: Wiley.

[23] Matouk, A.E., (2016), Chaos synchronization of a fractional-order modified Van der Pol-Duffing system via new linear control, backstepping control and Takagi-Sugeno fuzzy approaches, Complexity, 21, pp.116-124.

[24] Podlubny, I., (1999) Fractional Differential Equations, Academic Press, New York, NY, USA.

[25] Xu, S., (2014), Dynamics of a general prey–predator model with prey-stage structure and diffusive effects, Computers & Mathematics with Applications, 68, pp.405–423.

[26] Zhou, Y., Xiao, D., Li, Y., (2007), Bifurcations of an epidemic model with non-monotonic incidence rate of saturated mass action, Chaos, Solitons and Fractals, 32, pp.1903–1915.

Moustafa El-shahed is a Professor of mathematics at Qassim University, Saudi Arabia. El-shahed's academic background started in 1986 when he received his Bachelor of Science degree, from Ain Shams University, Egypt. In 1997 he earned his Master of Science degree in pure mathematics from the same university. In 2000, he received his Doctor of Science degree in mathematics from Ain Shams University, Egypt.

His research interests focus on differential equations, biomathematics, mathematical modeling, fractional calculus.

A. M. Ahmed is a Professor of Pure Mathematics in Department of Mathematics, Faculty of Science, Al-Azhar University, Cairo, Egypt. A. M. Ahmed, obtained his B.Sc. (1997), M.Sc. (2000) and Ph. D. (2004) from Al-Azhar University. He obtained the rank Associate Professor from Al-Azhar University in 2012 and the rank Professor from Al-Azhar University in January 2017. His research interests fokus on differential equations, dynamical systems and its applications.

Ibrahim M. E. Abdelstar is an Assistant Lecturer at Faculty of Business & Economics, Qassim University Saudi Arabia. Received the Master degree in pure mathematics at Faculty of science, Al-Azhar University. His research interests are in the areas of applied mathematics, economic dynamics, mathematical biology, chaos, nonlinear dynamical systems and fractional calculus.